Use WGH results for values of $2D_1/\lambda$ for which curves exist. These curves are given in Fig. 7 and in Fig. 6.1–10 of the WGH, where the appropriate changes in notation are the same as those indicated above for parameter d'. For all other values of $2D_1/\lambda$ and for $Z_{01}/Z_{02} \leq 1.0$, one can alternatively interpolate between the curves or employ the MRI theoretical expressions, appropriately modified. This modification consists of computing n via equivalence relation 1a) of Table III, where n' is given by (2), d'_{WGH} has been discussed above, d is obtained from relation 4a) of Table III, and X_a/Z_0 is given by (3). For $Z_{01}/Z_{02} > 1.0$, it is probably advisable to use WGH results as the MRI correction procedure was not checked in this range. Excellent agreement between the design and actual values should be obtained.

C. Parameter d—Reference Plane Shift in Main Arm

Here the recommendation to be made is a hybrid one. The IBM experimental data is considered the most reliable. The MRI experimental data appears to yield a value somewhat smaller than expected. For $Z_{01}/Z_{02} < 1.0$ and for all values of $2D_1/\lambda$, interpolate between IBM curves (extrapolate curves if necessary). For $Z_{01}/Z_{02} > 1.0$ and for all values of $2D_1/\lambda$ use "Modified MRI Theory," which requires that d be computed from equivalence relation 4a) of Table III, where X_a/Z_0 is given by (3). A fair prediction of the value of this parameter should be obtained by utilizing the above procedure except possibly in the particular region where $2D_1/\lambda > 0.7$ and $Z_{01}/Z_{02} > 1.0$. In this region only a helpful upper bound is available.

D. Parameter B—Shunt Susceptance

The curves obtained by the "Semi-Empirical Procedure" and plotted in Fig. 11 should be utilized to determine the value of the parameter B. This is also a hybrid recommendation as these curves were drawn by combining IBM experimental data for parameter d, for the range $Z_{01}/Z_{02} < 1.0$ and for all values of $2D_1/\lambda$, with "Modified MRI Theory." The latter requires the computation of B via equivalence relation 2a) of Table III, where the parameters occurring in this relation have been discussed above. The curves are to be taken as fairly reliable except for $2D_1/\lambda > 0.7$ and $Z_{01}/Z_{02} > 1.0$, where the experimental drop-off in d would indicate a lower value for B.

Acknowledgment

The authors are indebted to George F. Bland of IBM for his initiation of this study, his many helpful suggestions, and his continued interest and encouragement. They are also grateful to William R. Jones, formerly of IBM, who contributed substantially to this effort in its initial stages.

The Use of Exponential Transmission Lines in Microwave Components*

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Summary—This paper describes some techniques for utilizing exponential transmission lines in microwave components in order to reduce element lengths, and hence size and weight, and to significantly increase the operating frequency range. Formulas are developed which relate line length to the frequency and rate of taper for transmission line resonators, and a nomogram is included for easy determination of spurious frequencies. Additional formulas are given for the distributed representation of lumped elements using exponential sections of both coaxial and strip transmission line, and their use described in application to microwave filters and related components. In addition, the paper describes how unusually large rejection bandwidths can easily be obtained by proper selection of the individual element lengths and rates of taper.

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Introduction

In THE LAST few years there has been a growing awareness of the need for new designs of microwave components which combine the advantages of decreased size and weight, ease of fabrication, and extended coverage of the microwave spectrum. In this paper various techniques for designing microwave components using exponential transmission line sections will be presented and their advantages and limitations will be considered. In particular it will be shown that the use of exponential sections of strip transmission line in the design of microwave filters offers significant savings in volume and weight, variable form factors, greatly extended rejection bandwidths, and the same ease of construction as with other strip-line components. Formulas are developed which relate line length to the frequency and rate of taper for transmission line resonators, and a nomogram is included for rapid determination of spurious frequencies. Additional formulas are developed for the distributed approximation of lumped elements using exponential sections of line, and for the Q of resonators.

Techniques are also discussed for reducing the overall length of coaxial cavities commonly employed in RF amplifiers. It is shown that by approximating the exponential taper with a linear taper, and hence eliminating some of the difficulties in fabrication, reductions in length of 15–25 per cent are readily obtained. The transformer properties of exponential lines will not be considered.

General Transmission Line Equations

For the TEM mode of propagation the differential equations representing the voltage $V(x)$ and current $I(x)$ for nonuniform transmission lines are

$$\begin{aligned} dV/dx + Z(x)I &= 0, \\ dI/dx + Y(x)V &= 0, \end{aligned} \tag{1}$$

where $Z(x)$ and $Y(x)$ are the series impedance and shunt admittance per unit length of line, respectively, and they are arbitrary and continuous functions of the position x along the line.

Differentiation of (1) yields the familiar set of second-order linear differential equations

$$\begin{aligned} d^2V/dx^2 - (1/Z)(dZ/dx)(dV/dx) - YZV &= 0 \\ d^2I/dx^2 - (1/Y)(dY/dx)(dI/dx) - YZI &= 0. \end{aligned} \tag{2}$$

Sugai[1] has shown that these second-order linear differential equations yield a general Riccati's differential equation

$$dr/dx + P_1(x)r + Q_1(x)r^2 = Q_1(x), \tag{3}$$

where r is the reflection coefficient defined by

$$r(x) = \frac{V - IK_0(x)}{V + IK_0(x)}, \tag{4}$$

and

$$\begin{aligned} K_0(x) &= \sqrt{Z(x)/Y(x)} \\ P_1(x) &= -2\sqrt{Z(x)Y(x)} \\ Q_1(x) &= -\frac{1}{2K_0(x)}(dK_0/dx). \end{aligned} \tag{5}$$

There is much in the literature[2] pertaining to both approximate and exact solutions for nonuniform transmission lines. A method recently proposed[3] utilizes two newly discovered transforms to reduce (3) to a first-order linear differential equation.

This paper is concerned more specifically with the exponential type of nonuniform transmission line which is defined by

$$K_0(x) = K_{01}e^{\delta x}; \qquad \delta \geq 0, \tag{6}$$

where K_{01} is the impedance level of the line at $x=0$, *i.e.*, the low-impedance end, and δ is the rate of taper.

Combining (3) and (6), and assuming negligible dissipation ($R=G=0$), the differential equation for an exponential transmission line is

$$dr/dx - 2\gamma r + (\delta/2)(1 - r^2) = 0, \tag{7}$$

whose solution is easily found to be

$$r(x) = 2\gamma/\delta + A\,\frac{1 - B\tanh(\delta Ax/2)}{\tanh(\delta Ax/2) - B}, \tag{8}$$

where

$$\begin{aligned} A^2 &= 1 + 4\gamma^2/\delta^2 \\ B &= \frac{A - (r_0 - 2\gamma/\delta)\tanh(\delta Al/2)}{A\tanh(\delta Al/2) - (r_0 - 2\gamma/\delta)} \\ \gamma &= j\beta = j(2\pi/\lambda) \end{aligned} \tag{9}$$

and $r(l)=r_0$ is the value of the reflection coefficient evaluated from the boundary conditions.

Exponential Transmission Line Resonator

It will now be shown that the particular nonuniform lossless transmission line presently under discussion can support standing waves just as does the ordinary uniform line, and that definite practical advantages such as shorter lengths and better form factors are obtainable. The type of terminations commonly employed to cause complete reflection consist of the open circuit, short circuit, and pure reactance. In each case $|r_0| = +1$, so that subject to the restrictions imposed by the solution of (8), the exponential line should perform as a resonator.

Consider an antiresonant[4] section of short-circuited exponential line. This will require a short circuit at the position $x=l$ as shown in Fig. 1(a). Under these circumstances,

$$r_0 = -1, \qquad r(0) = r(x)_{x=0} = +1, \tag{10}$$

so that from (8)

$$\tanh(1 + 4\gamma^2/\delta^2)^{1/2}(\delta l/2) = (1 + 4\gamma^2/\delta^2)^{1/2}. \tag{11}$$

[1] I. Sugai, "The solutions for nonuniform transmission line problems," Proc. IRE, vol. 48, pp. 1489–1490; August, 1960.
[2] H. Kaufman, "Bibliography of nonuniform transmission lines," IRE Trans. on Antennas and Propagation, vol. AP-3, pp. 218–220; October, 1955.
[3] I. Sugai, "A new exact method of nonuniform transmission lines," Proc. IRE, vol. 49, pp. 627–628; March, 1961.

[4] An antiresonant line shall be defined as one whose sending-end impedance is infinite when its output is suitably terminated so as to cause total reflection of the incident wave. It may be compared to a lumped LC parallel circuit at resonance. Similarly, a resonant line shall be defined as one whose sending-end impedance is zero for the same load terminal conditions as above. It may be compared to a lumped LC series circuit at resonance.

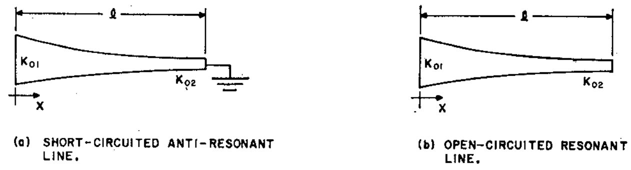


Fig. 1—Types of short- and open-circuited lines required for antiresonance and resonance, respectively (shown for strip-line).

Denoting the length of line required for antiresonance by l_∞ and making use of the fact that $\gamma = j\beta$, (11) yields

$$l_\infty = \frac{n\pi + \tan^{-1}(4\beta^2/\delta^2 - 1)^{1/2}}{\beta(1 - \delta^2/4\beta^2)^{1/2}}, \quad n = 0, 1, 2, \cdots \quad (12)$$

and

$$l_{\infty,\,\min} = \frac{\frac{2}{\pi}\tan^{-1}(4\beta^2/\delta^2 - 1)^{1/2}}{(1 - \delta^2/4\beta^2)^{1/2}}\left(\frac{\lambda}{4}\right). \quad (13)$$

Similarly, in the case of the resonant section of open-circuited exponential line shown in Fig. 1(b),

$$r(0) = r_0 = +1 \quad (14)$$

whereupon, if l_0 denotes the length of line required for resonance, one finds

$$l_0 = \frac{(n+1)\pi - \tan^{-1}(4\beta^2/\delta^2 - 1)^{1/2}}{\beta(1 - \delta^2/4\beta^2)^{1/2}}, \quad n = 0, 1, 2, \cdots \quad (15)$$

and

$$l_{0,\,\min} = \frac{1 - \frac{1}{\pi}\tan^{-1}(4\beta^2/\delta^2 - 1)^{1/2}}{(1 - \delta^2/4\beta^2)^{1/2}}\left(\frac{\lambda}{2}\right). \quad (16)$$

The corresponding lengths of open- and short-circuited sections of exponential line required to produce antiresonance and resonance, respectively, are equal and given by

$$l = \frac{(n+1)\pi}{(1 - \delta^2/4\beta^2)^{1/2}}\,\frac{1}{\beta}, \quad n = 0, 1, 2, \cdots \quad (17)$$

and

$$l_{\min} = \frac{\lambda/2}{(1 - \delta^2/4\beta^2)^{1/2}}. \quad (18)$$

The significance of these results can be appreciated upon examination of Fig. 2 wherein (13) and (16) are plotted. In the case of resonators corresponding to curve b of the figure, for example, it is theoretically possible to obtain elements of length v/ω for values of $\delta/\beta = 2$, or approximately 37 per cent shorter than the corresponding uniform line resonator at the same frequency. This is a conservative figure, however, since the wider line widths at the open-circuited ends of the resonators result in pronounced fringing of the electric field.

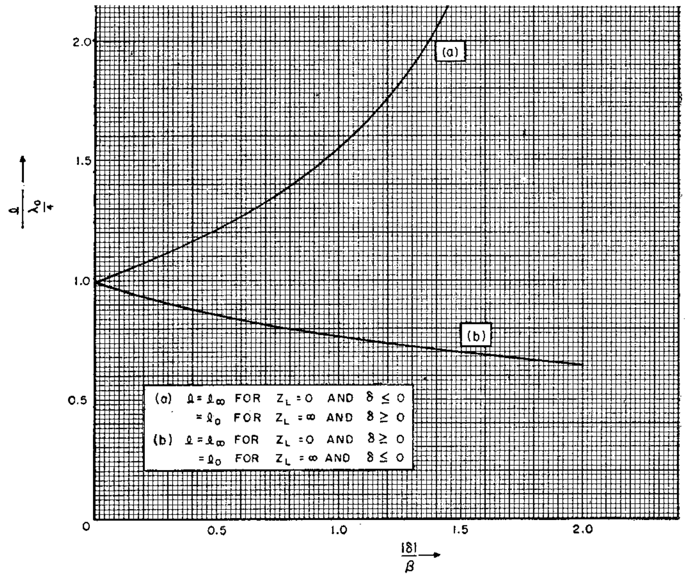


Fig. 2—Minimum lengths of open- and short-circuited sections of transmission line required for resonance and antiresonance, respectively.

This is, in effect, equivalent to a capacitive loading of the ends of the resonators, the result being that the electrical lengths of the lines are increased.

Calculations of the discontinuity effects of abruptly-ended center conductors in strip-line have been given by Altschuler and Oliner.[5] When applied to the open ends of exponential strip-line resonators, further reductions in length of approximately 2–10 per cent are easily obtained. For example, the length of an exponential transmission line resonator constructed in strip-line was 40 per cent shorter in length than the corresponding uniform line resonator at the same frequency, and there was no apparent degradation of Q-factor.

Exponential transmission line resonators can be usefully employed in many microwave strip-line components where decreased size and weight, ease of fabrication, and extended coverage of the microwave spectrum are important. The latter requirement, for example, is expressed in the manner by which the designer has control over the location of annoying spurious responses. Whereas the spurious responses of uniform transmission line resonators occur for frequencies which are integer multiples of the frequency of resonance, they are not so determined in the case of exponential transmission line resonators.

Examination of (13) shows the manner in which the spurious frequencies of exponential resonators are related to the ratio δ/β, or alternatively, to the length and rate of taper of the line. For convenience and as an aid to the designer, the relationship of the first spurious frequency f_1 to the resonant frequency f_0 and the physical constants of the line for two practical resonators is given

[5] H. M. Altschuler and A. A. Oliner, "Discontinuities in the center conductor of symmetric strip transmission line," IRE Trans. on Microwave Theory and Techniques, vol. MIT-8, pp. 328–339; May, 1960.

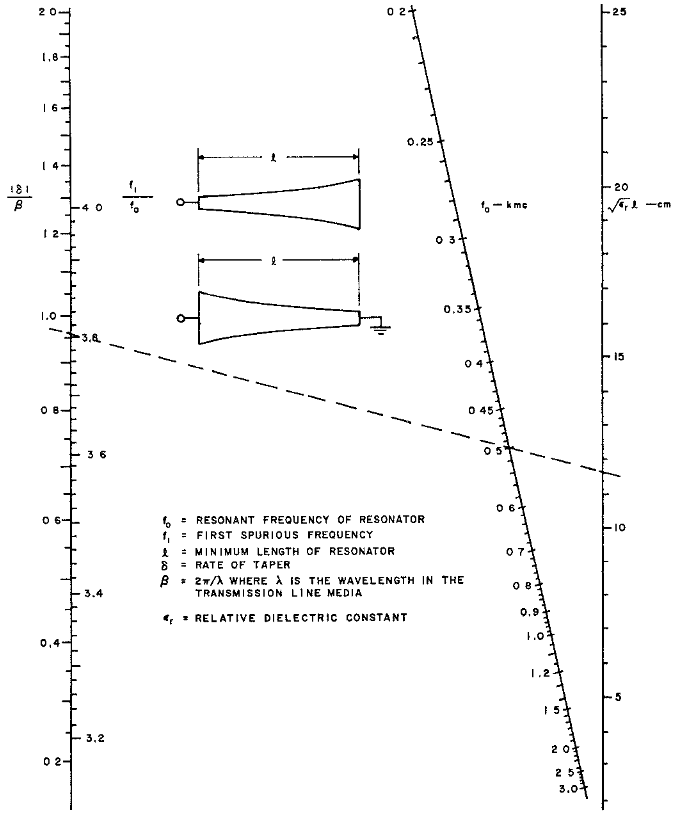
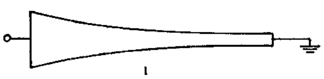
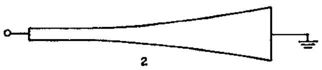
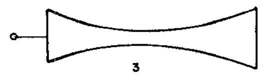
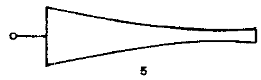


Fig. 3—Nomogram for determining spurious frequencies of an antiresonant and resonant resonator.

TABLE I

INPUT ADMITTANCE NEAR RESONANCE AND MINIMUM RESONANT LENGTH FOR VARIOUS TYPES OF ANTI-RESONANT LINES

ANTI RESONANT LINES	MINIMUM RESONANT LENGTH
Y_{S1} →  1	$\frac{S/\pi}{Q_0}\left(\frac{\lambda_0}{2}\right)$
Y_{S2} →  2	$\frac{1-S/\pi}{Q_0}\left(\frac{\lambda_0}{2}\right)$
Y_{S1} →  3	$\frac{S/\pi}{Q_0}\lambda_0$
Y_{S2} →  4	$\frac{1-S/\pi}{Q_0}(\lambda_0)$
Y_{S1} →  5	$\lambda_0/2Q_0$
Y_{S2} →  6	
$Y_{S1} = -j(1/K_{01})\left[Q\cot\left(RS(\omega/\omega_0)\right) - \delta/2\beta\right]$ $Y_{S2} = -j(1/K_{02})\left[Q\cot\left(RS(\omega/\omega_0)\right) + \delta/2\beta\right]$	$Q = (1-\delta^2/4\beta^2)^{1/2}$ $Q_0 = Q(\omega_0)$ $R = \frac{(1-\delta^2/4\beta^2)^{1/2}}{(1-\delta^2/4\beta_0^2)^{1/2}}$ $S = \tan^{-1}(4\beta_0^2/\delta^2-1)^{1/2}$

in the nomogram of Fig. 3. The advantages of using exponential resonators in microwave filters will become apparent in the next section.

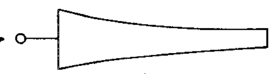
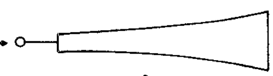
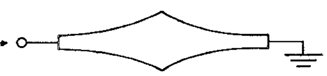
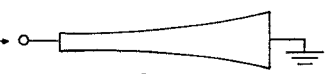
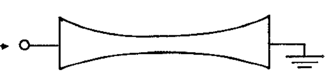
Exponential transmission line resonators can also be employed in the coaxial cavities of RF amplifiers. Exact exponential tapers for coaxial lines are difficult to construct. Consequently the advantages of shorter resonator lengths, and hence smaller cavity size and weight, may not justify the increased production difficulties. However, by approximating the exponential taper with a linear taper, reductions in resonator lengths of approximately 15–25 per cent are easily obtainable. The resultant ease of fabrication is evident.

In addition to decreasing the over-all length of transmission line resonators, it is also possible to extend their length. For example, if the rate of taper of the exponential resonator of Fig. 1(a) is negative, *i.e.*, the line is divergent, the length of the resonator is longer than the corresponding length of uniform line resonator at the same frequency and is given by (16).[6] This is particularly important at microwave frequencies where the lengths of uniform resonators become quite small and their fabrication in many instances unusually difficult. Tables I and II list additional types of resonant and antiresonant sections of exponential transmission line.

[6] The dependence of the resonant length upon the sign of δ may be seen upon examination of (11).

TABLE II

INPUT ADMITTANCE NEAR RESONANCE AND MINIMUM RESONANT LENGTH FOR VARIOUS RESONANT LINES

RESONANT LINES	MINIMUM RESONANT LENGTH
Y_{S3} →  1	$\frac{1-S/\pi}{Q_0}\left(\frac{\lambda_0}{2}\right)$
Y_{S4} →  2	$\frac{S/\pi}{Q_0}\ \frac{\lambda_0}{2}$
Y_{S4} →  3	$\frac{S/\pi}{Q_0}\lambda_0$
Y_{S3} →  4	$\lambda_0/2Q_0$
Y_{S4} →  5	
Y_{S3} →  6	$\frac{1-S/\pi}{Q_0}(\lambda_0)$
$Y_{S3} = j(1/K_{01})\left[Q\cot\left(RS(\omega/\omega_0)\right) + \delta/2\beta\right]^{-1}$ $Y_{S4} = j(1/K_{02})\left[Q\cot\left(RS(\omega/\omega_0)\right) - \delta/2\beta\right]^{-1}$	$Q = (1-\delta^2/4\beta^2)^{1/2}$ $Q_0 = Q(\omega_0)$ $R = \frac{(1-\delta^2/4\beta^2)^{1/2}}{(1-\delta^2/4\beta_0^2)^{1/2}}$ $S = \tan^{-1}(4\beta_0^2/\delta^2-1)^{1/2}$

Application to Filter Design

Much of the present day microwave filter design is based upon the approximate realization of a lumped element design. This requires the judicious selection of transmission line structures which exhibit the characteristics of the lumped elements. In this regard one can realize many structures equivalent to lumped elements, but in general their selection is governed by many practical considerations. Specifically, the type of transmission line in which the filter is to be constructed, the frequency band required to yield a valid approximation, and the available volume and hence the physical size of the elements are contributing factors. In addition, the realization of distributed constant filters having sufficient rejection bandwidths will enter into the choice of possible structures. For example, in the design of bandpass filters the elimination of the spurious responses, and hence the attainment of large rejection bandwidths, is usually accomplished by connecting in cascade with the filter an additional low-pass filter whose cutoff frequency lies above the pass band and below the first spurious response.

Examination of (13) and (16) suggests that the lumped constants of the prototype filter can be approximated using appropriate sections of exponential transmission line, and that the element lengths will be significantly increased or decreased in comparison to those obtained when only uniform lines are employed. Moreover, the control of the location of the spurious responses, as suggested upon examination of Fig. 3, indicates the possibility of obtaining unusually large rejection bandwidths. The following sections will serve to exploit these possibilities further.

Short- and Open-Circuited Exponential Lines

One familiar technique for approximating the characteristics of a lumped reactance is to utilize appropriate sections of short-circuited and open-circuited transmission line. A short length of short-circuited transmission line can be made to approximate the behavior of a lumped inductance, whereas a similar length of open-circuited line can approximate a lumped capacitor. It will now be shown that similar techniques utilizing short sections of open- and short-circuited exponential line can be employed, but with the resultant advantages of reduced lengths, and in most cases, better form factors.

The solutions of (1) for $V(x)$ and $I(x)$ are found to be[7]

$$V(x) = a_1e^{\Gamma x} + a_2e^{-\Gamma x}$$
$$I(x) = b_1e^{\Gamma x} + b_2e^{-\Gamma x} \qquad (19)$$

where

$$\Gamma^2 = \gamma^2 + (\delta/2)^2 \qquad (20)$$

and, assuming a transmission line with negligible dissipation ($R=G=0$), $\gamma^2 = -\beta^2$.

The input impedance of an exponential line of length l terminated in an arbitrary load impedance Z_L is

$$Z_s = \frac{K_{01}K_{02}}{Z_L}\,\frac{1+\dfrac{Z_L}{K_{02}}\left[\sqrt{1-\delta^2/4\beta^2}-j(\delta/2\beta)\right]+e^{-2\Gamma l}\left[\dfrac{Z_L}{K_{02}}\sqrt{1-\delta^2/4\beta^2}+j\dfrac{\delta}{2\beta}-1\right]}{1+\dfrac{K_{02}}{Z_L}\left[\sqrt{1-\delta^2/4\beta^2}+j(\delta/2\beta)\right]+e^{-2\Gamma l}\left[\dfrac{K_{02}}{Z_L}\sqrt{1-\delta^2/4\beta^2}-j\dfrac{\delta}{2\beta}-1\right]}$$

$$= \frac{K_{01}K_{02}}{Z_L}\,\frac{\dfrac{Z_L}{K_{02}}\sqrt{1-\delta^2/4\beta^2}+(1-j\delta Z_L/2\beta K_{02})\tanh \Gamma l}{\dfrac{K_{02}}{Z_L}\sqrt{1-\delta^2/4\beta^2}+(1+j\delta K_{02}/2\beta Z_L)\tanh \Gamma l}, \qquad (21)$$

where K_{01} and K_{02} are the values of $K_0(x)$ evaluated at $x=0$ and $x=l$, respectively.

If the exponential line defined by (6) is terminated in a load impedance $Z_L=0$, (21) reduces to

$$Z_s = jX$$
$$= jK_{01}\frac{1}{\sqrt{1-\delta^2/4\beta^2}\cot \beta\sqrt{1-\delta^2/4\beta^2}l - \delta/2\beta}. \qquad (22)$$

When the section of exponential line is open-circuited ($Z_L=\infty$), the input impedance becomes

$$Z_s = \frac{1}{jB} = -jK_{01}$$
$$\cdot\left[(\delta/2\beta)+\sqrt{1-\delta^2/4\beta^2}\cot \beta\sqrt{1-\delta^2/4\beta^2}l\right]. \qquad (23)$$

[7] J. J. Karakash, "Transmission Lines and Filter Networks," The Macmillan Co., New York, N. Y., p. 140; 1950.

For short sections of line the slopes of the functions (22) and (23) can be approximated respectively by

$$dX/d\omega = K_{01}\frac{l/v}{1-\delta l/2}, \tag{24}$$

$$dB/d\omega = \frac{1}{K_{01}}\frac{l/v}{1+\delta l/2}, \tag{25}$$

since $\beta = \omega/v$.

Comparing (24) and (25) to the slopes of the reactance and susceptance functions for a lumped inductance and capacitance, respectively, one obtains

$$L = K_{01}\frac{l/v}{1-\delta l/2}$$

$$-2 \le \delta/\beta \le +2$$

$$C = \frac{1}{K_{01}}\frac{l/v}{1+\delta l/2}.$$

Thus, short sections of short- and open-circuited exponential transmission line can be used to approximate an inductance and capacitance. The distributed representation of lumped elements is shown in Fig. 4.

Low-Q Resonant Circuits

In many applications there is often the need to approximate low-Q lumped resonant circuits. This can easily be accomplished by using exponential transmission line sections with values of the ratio $\delta/\beta = 2$, in which case the resultant line lengths may be significantly reduced. For example, a parallel resonant circuit in shunt with the line can be approximated with two sections of exponential line, one open-circuited and one short-circuited, as illustrated in Fig. 5(a). For this equivalence to be valid the lengths of the two stubs are found to be

$$l_C = \frac{2/\delta}{1+\dfrac{\sqrt{L/C}}{K_{01}}}$$

$$l_L = \frac{2/\delta}{1+\dfrac{K_{01}}{\sqrt{L/C}}} \tag{28}$$

and

$$l_T = l_L + l_C = 2/\delta = 1/\beta_0, \tag{29}$$

where C and L are the shunt capacitance and shunt inductance of the lumped resonant circuit, respectively, and K_{01} is the impedance level of the shorted exponential line section at its low-impedance end.

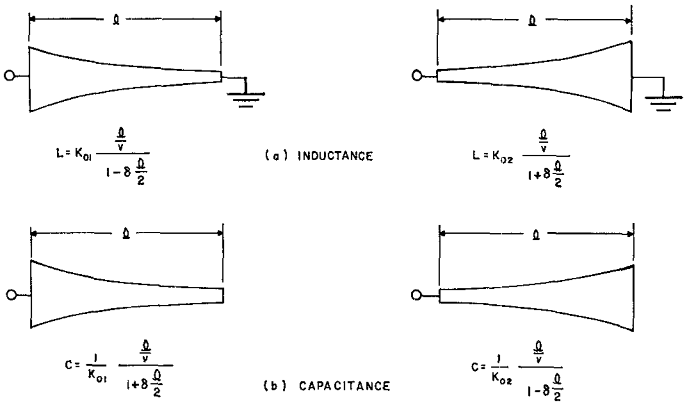


Fig. 4—Distributed representation of lumped elements (shown for strip-line).

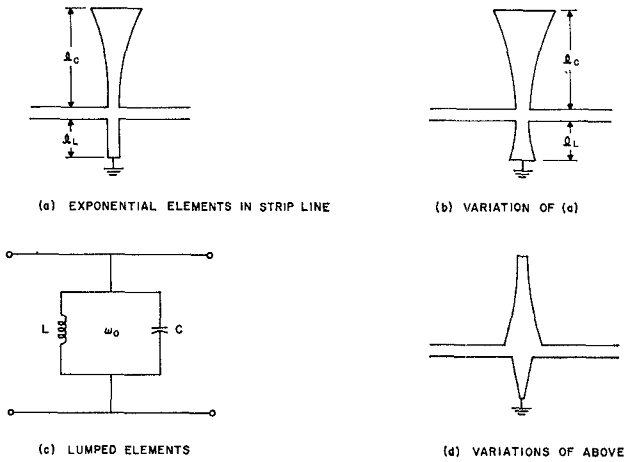


Fig. 5—Distributed exponential transmission line representation of low-Q lumped resonant circuits ($|\delta/\beta| = 2$).

In many applications involving shorted line stubs it may happen that the element lengths are too short to permit an exact determination of the position of the physical short circuit. This is especially true for strip line construction. One technique for eliminating this indeterminacy is to use the configuration shown in Fig. 5(b). There it is seen that the choice of a negative value of δ for the short-circuited stub yields an element length which is significantly increased over that of the uniform line stub. This permits a reduction in those errors associated with the smaller elements. For this configuration the element lengths are

$$l_L = \frac{2/\delta}{\dfrac{K_{02}}{\sqrt{L/C}}-1}$$

$$l_C = \frac{2/\delta}{\dfrac{\sqrt{L/C}}{K_{02}}+1} \tag{30}$$

and

$$l_T = l_L + l_C = \frac{\left[\frac{K_{02}}{\sqrt{L/C}}\right]^2 + 1}{\left[\frac{K_{02}}{\sqrt{L/C}}\right]^2 - 1}\left(\frac{2}{\delta}\right), \quad (31)$$

where K_{02} is the impedance level of the short-circuited stub at its high-impedance end. Similar results are easily obtained for the configuration shown in Fig. 6(d).

Arbitrarily Terminated Sections of Exponential Line

The occasion often arises in practice when one is interested in the input impedance of a section of transmission line of specified length which is terminated in relatively large, or small, impedances. The inversion properties of $\lambda/4$ resonators constructed with uniform transmission line, for example, are well known and some of their applications have been discussed by various authors.[8,9] It will now be shown that similar properties are obtained with exponential line.

With reference to (21) and Fig. 6(a) the input impedance of a section of exponential line of length $l_{\infty,\min}$ normalized with respect to the impedance level at the input, and terminated in an impedance Z_L, is

$$\frac{Z_s}{K_{02}} = \frac{\frac{Z_L}{K_{01}}\left[1 - (|\delta|/2\bar{\beta})G(\omega)\right] + j(\beta/\bar{\beta}_0)G(\omega)}{\left[1 + (|\delta|/2\bar{\beta})G(\omega)\right] + j(Z_L\beta/K_{01}\bar{\beta})G(\omega)}, \quad (32)$$

where

$$G(\omega) = \tan\left[\frac{\bar{\beta}}{\bar{\beta}_0}\tan^{-1}\sqrt{4\beta_0^2/\delta^2 - 1}\right] \quad (33)$$

and

$$\bar{\beta} = \sqrt{1 - \delta^2/4\beta^2}$$
$$\bar{\beta}_0 = \beta_0\sqrt{1 - \delta^2/4\beta_0^2}. \quad (34)$$

For $Z_L/K_{02} \gg 1$, (32) simplifies to

$$Z_s/K_{02} = jX(\omega)/K_{02} + \bar{Z}/K_{02}, \quad (35)$$

where

$$X(\omega) = K_{02}\frac{\frac{|\delta|}{2\bar{\beta}}G(\omega) - 1}{\frac{\beta}{\bar{\beta}}G(\omega)} = K_{02}F(\omega) \quad (36)$$

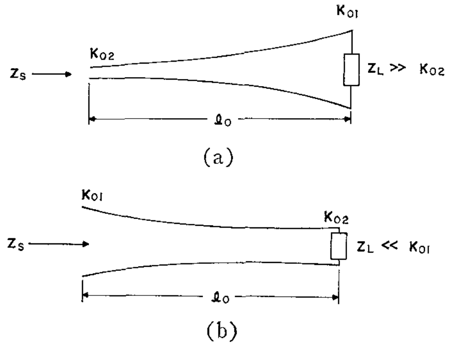


Fig. 6—Exponential transmission line circuits which, for the terminal conditions indicated, possess important inversion properties as described in the text.

and

$$\bar{Z} = K_{01}K_{02}/Z_L. \quad (37)$$

Similarly, the input impedance of the section of exponential line of length $l_{\infty,\min}$ shown in Fig. 6(b) when terminated in an impedance $Z_L \ll K_{01}$, is

$$Z_s/K_{01} = \frac{1}{jK_{01}B(\omega) + K_{01}\bar{Y}}, \quad (38)$$

where

$$\bar{Y} = Z_L/K_{01}K_{02} \quad (39)$$

and

$$B(\omega) = \frac{1}{K_{01}}\frac{\frac{|\delta|}{2\bar{\beta}}G(\omega) - 1}{\frac{\beta}{\bar{\beta}}G(\omega)} = F(\omega)/K_{01}. \quad (40)$$

Examination of (35) and (38) indicates that the impedance Z_L at the load end of the resonators is reflected toward the input, and inverted with respect to the product of the high- and low-impedance levels of the line.[10] Compare this to a $\lambda/4$ section of uniform line which, when terminated in an appropriate load impedance under similar conditions, is inverted with respect to the square of the characteristic impedance Z_0.

The function $F(\omega)$ is seen to depend upon the rate of taper of the exponential line in such a manner as to make the expressions (36) and (40) too unwieldy. For convenience $F(\omega)$ is shown in Fig. 7 for typical values of the ratio δ/β_0. For frequencies near ω_0 the approximation

$$F(\omega) = m\frac{\omega - \omega_0}{\omega_0}, \quad (41)$$

where

[8] S. B. Cohn, "Direct-coupled-resonator filters," Proc. IRE, vol. 45, pp. 187–196; February, 1957.

[9] S. B. Cohn, "Parallel-coupled transmission-line-resonator filters," IRE Trans. on Microwave Theory and Techniques, vol. MIT-6, pp. 223–231; April, 1958.

[10] This property of the exponential transmission line can be used to advantage for the design of band-pass microwave filters in order to obtain shorter element lengths, greater rejection bandwidths, and controllable form factors. A paper is presently being readied by this author which presents the design formula for a class of strip-line band-pass filters having exponential transmission line elements.

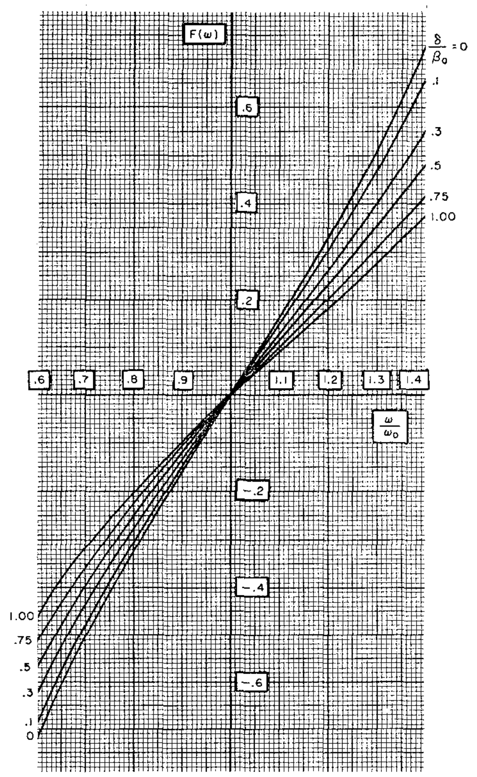


Fig. 7—Variation of $F(\omega)$ with frequency for various values of the ratio δ/β_0.

$$m = \frac{\tan^{-1}\sqrt{4/C^2 - 1}}{\sqrt{1 - C^2/4}} \csc^2 (\tan^{-1}\sqrt{4/C^2 - 1}) + \frac{2/C}{1 - 4/C^2} \tag{42}$$

and

$$C = |\delta| / \beta_0 \tag{43}$$

is quite accurate up to and including bandwidths of 30 per cent.

Q of the Exponential Resonator

The calculation of the Q of an exponential resonator has previously been presented,[11] and the result expressed in the form of an infinite series. It is the purpose of the following discussion to obtain an expression for Q which is in closed form, and hence more amenable to calculation.

For any mode the Q of a resonator is uniquely defined by the expression

$$Q = \frac{\omega_0 U}{W_L} = \omega_0 \frac{\iiint \frac{\mu}{2} |H|^2 dV}{\iint \frac{R_s}{2} |H_t|^2 dS} \tag{44}$$

[11] R. N. Ghose, "Exponential transmission lines as resonators and transformers," IRE Trans. on Microwave Theory and Techniques, vol. MIT-5, pp. 213–217; July, 1957.

assuming that there are no losses due to the dielectric material filling the volume of the cavity. The numerator integral of (44) represents the energy storage in the magnetic fields at the instant when these are a maximum, and the denominator integral represents the total of all power losses in the walls. In order to evaluate (44) an expression for the magnetic field must first be found.

For an exponential transmission line short-circuited at the position $x = l$, the current at any point along the line may be found from (19) and (20) to be

$$|I(x)| = \frac{A}{\beta K_{01}} e^{-(\delta/2)x} \left[\frac{\delta}{2} \sin (\bar{\beta} l - \bar{\beta} x) - \bar{\beta} \cos (\bar{\beta} l - \bar{\beta} x) \right], \tag{45}$$

where A is a constant. For the TEM mode the magnetic field is given by

$$|H_\phi| = \frac{1}{2\pi r} |I(x)|. \tag{46}$$

For the present discussion let it be assumed that the radius of the outer cylinder of the coaxial resonator is a fixed constant b, and that of the center cylinder is $a(x)$. This is the situation most often encountered in practice. Under these conditions the stored magnetic energy of the resonator is

$$U_m = \frac{\epsilon \eta^2 A^2}{2\pi^2 \beta^2 K_{01}^2} \int_0^l \int_0^{2\pi} \int_{a(x)}^b \frac{e^{-\delta x}}{r^2} \cdot \left[\frac{\delta}{2} \sin (\bar{\beta} l - \bar{\beta} x) - \bar{\beta} \cos (\bar{\beta} l - \bar{\beta} x) \right]^2 r dr d\phi dx$$
$$= \frac{2\epsilon\eta A^2}{\beta^2 K_{01}} \left[\frac{\delta^2}{8} \left(l - \frac{1}{2\bar{\beta}} \right) + \frac{\bar{\beta}^2 l}{2} - \frac{\delta}{4} + \left(\frac{\delta^2}{16\bar{\beta}} + \frac{\delta}{4} \right) \cos 2\bar{\beta} l + \frac{\bar{\beta}}{4} \sin 2\bar{\beta} l \right]. \tag{47}$$

The length of a short-circuited resonant line will be given by (18). For this case (47) reduces to

$$U_m = \frac{A^2 l}{v K_{01}} \cdot \tag{48}$$

In order to determine the total power losses in the walls, it is necessary to integrate the tangential magnetic field over all internal surfaces of the cavity. That is,

$$W_L = \frac{R_s}{2} \int_0^l \int_0^{2\pi} a(x) \left| H_\phi \right|^2_{r=a(x)} d\phi dx + \frac{R_s}{2} \int_0^l \int_0^{2\pi} b \left| H_\phi \right|^2_{r=b} d\phi dx + \frac{R_s}{2} \int_0^{2\pi} \int_{a(0)}^b \left| H_\phi \right|^2_{x=0} r dr d\phi + \frac{R_s}{2} \int_0^{2\pi} \int_{a(l)}^b \left| H_\phi \right|^2_{x=l} r dr d\phi.$$

Performing the indicated integrations the total power loss of the cavity, for a cavity length given by (18), is

$$
\begin{aligned}
W_L &= \frac{A^2R_s}{4\pi\beta^2K_{01}{}^2}\left\{4\bar{\beta}^2\left(e^{-(\delta/2)l}\ln\frac{b}{a(l)}+\ln\frac{b}{a(0)}\right)\right.\\
&\quad+\frac{1}{a(0)}\left(\frac{1-e^{-\gamma l}}{1+\gamma^2/4\bar{\beta}^2}\right)\left[\frac{2\bar{\beta}^2}{\gamma}\left(1+\frac{\delta^2}{4\bar{\beta}^2}\right)+\gamma+\delta\right]\\
&\quad\left.+\frac{1}{b}\left(\frac{1-e^{-\delta l}}{1+\delta^2/4\bar{\beta}^2}\right)\left[\frac{2\bar{\beta}^2}{\delta}\left(1+\frac{\delta^2}{4\bar{\beta}^2}\right)+2\delta\right]\right\}\\
&= \frac{A^2R_s}{4\pi\beta^2K_{01}{}^2}F(\delta) \qquad (49)
\end{aligned}
$$

where

$$\gamma = \frac{2\pi}{\eta l}(K_{01}-K_{02})+\delta$$

and δ is the rate of taper defined earlier. The present definition of γ should not be confused with that given in (9).

The Q-factor for an exponential coaxial resonator is then

$$Q = \frac{\omega_0 U_m}{W_L} = \frac{8\pi^2\beta^2 l K_{01}}{\lambda R_s}F^{-1}(\delta). \qquad (50)$$

The exponential resonator should reduce to a uniform line resonator when the rate of taper $\delta=0$. For this condition $l=\lambda/2$, and $K_{01}=K_0$, the characteristic impedance of the uniform line. Then

$$F(0) = \beta^2\left(8\ln\frac{b}{a}+\frac{\lambda}{a}+\frac{\lambda}{b}\right)$$

so that

$$Q = \frac{4\pi^2K_0}{R_s}\left(8\ln\frac{b}{a}+\frac{\lambda}{a}+\frac{\lambda}{b}\right)^{-1}$$

which is the Q for a uniform coaxial line resonator one half-wavelength long.

Experimental Results

A series of short-circuited antiresonant exponential resonators were constructed and evaluated in order to test the validity of the theoretical results before and after suitable compensation for the end effects discussed earlier in the text. The resonators were constructed of strip line and designed to resonate at 500 Mc. Measured values of the resonant frequency differed from calculated values by as much as 8 per cent before any correction for end effects was made, and less than 2 per cent after correction. The results of these measurements are shown in Fig. 8.

A simple band-pass filter was fabricated in order to demonstrate the practicality of utilizing exponential transmission line elements in microwave filter designs.

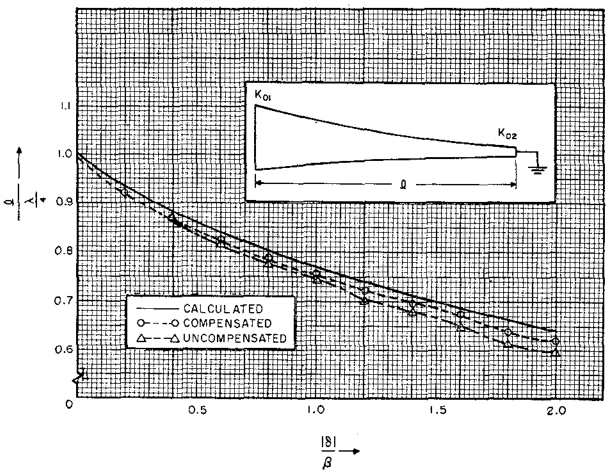


Fig. 8—Effects of compensation on the resonant length of an exponential resonator.

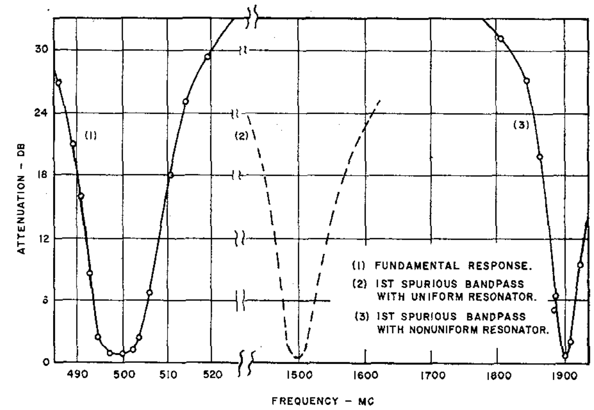


Fig. 9—Frequency response of the experimental filter ($\delta/\beta_0=1$).

The filter consisted of two magnetically coupled short-circuited elements which were designed for antiresonance at a frequency of 500 Mc. In addition, the condition was imposed that the first spurious responses of both elements should occur at $3.8f_0$. This fixed the value of δ and l as determined from the nomogram of Fig. 3. The dashed line indicated on the figure illustrates the use of the nomogram for the present case.

Fig. 9 shows the results of the experimental filter. It is seen that the measured rejection bandwidth agrees quite well with the anticipated value, and is 30 per cent greater than that obtained when using uniform line resonators. This suggests the possibility that, by selecting the nonuniform transmission line elements so that their individual spurious responses occur at different frequencies, it should be possible to design microwave filters which possess extremely wide rejection bandwidths.

Acknowledgment

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